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STOCHASTIC LANCHESTER-TYPE
COMBAT MODELS I

by

L. BILLARD

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STOCHASTIC LANCHESTER-TYPE COMBAT MODELS I

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ABSTRACT

There has been a lot of activity in recent years on the study of Lanchester-type combat models, especially from a deterministic standpoint. We consider some of these models in a stochastic framework and indicate how the appropriate deterministic model can be recast stochastically. Techniques for obtaining the corresponding solutions to the resultant differential-difference equations are discussed. These techniques are similar to those developed for use in other population process modelling situations such as epidemic theory and competition models. These are used to give the actual solution for the stochastic model of the original Lanchester model.

KEYWORDS: Lanchester, deterministic, stochastic, state probabilities, duration time.

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1. INTRODUCTION

Since World War II there has been much activity on the study of combat models based on the original work of Lanchester (1914). The so-called Lanchester models describe the situation in which two forces are in combat with each other with each side losing men or equipment (tanks) by attrition in accordance with some preassigned attrition law. These laws vary from the very simplest formulation used by Lanchester to quite complex cases. An extensive review of these different situation is presented in Taylor (1978).

Basically the model for the combat process can be described in terms of the sizes of the two combat forces. Most of the work in the literature so far has confined its attention to studying the combat process in a deterministic framework. While such an approach can prove useful in providing broad guidelines as to the behavior of a given combat situation, it is likely that a more accurate account can be obtained when the process is viewed stochastically. Therefore, in this paper it is shown how analogous stochastic combat models can be developed, and the appropriate solutions are derived. Specifically we show that a stochastic analogue of the Lanchester-type combat models is nothing but a particular bivariate death process.

Furthermore if we wish to extend the simplest combat models to allow for reinforcements we will have a bivariate birth and death process.

In the following section, we describe the general bivariate pure death process. This is applied to the particular case of the original Lanchester model in Section 3. In Section 4, we show how several other models can be formulated as a bivariate death process. These are models which generally have been considered in the literature from a deterministic approach. Some have been fully solved and others only partially solved (that is, for some finite subset of the governing parameters). We note that using the results of our Section 2, formal solutions can be found to the stochastic analogues in all cases. Then, in Section 5, we consider some models whose stochastic formulation is that of a bounded bivariate birth and death process. Finally we consider combat duration time in Section 6. Numerical results and comparisons for varying parameter values and battle force sizes will be presented in a companion paper.

2. BIVARIATE DEATH MODEL

Let $B(t)$ and $R(t)$ be the size of the Blue and Red forces at time t, respectively, and let particular state values be b and r, respectively. Typically, we may be interested in the number of men, or of tanks, or of both men and tanks, etc. We interpret "size" as the number of units (men, tanks, etc.) under discussion. Suppose that at time $t = 0$, $B(0) = B_0$ and $R(0) = R_0$.

The two forces are in combat with each other and units are lost according to various attrition laws, the particular attrition law depending on the actual modelling situation at hand. For general attrition rates $\alpha(b,r,t)$ and $\beta(b,r,t)$ for the Blue and Red forces, respectively, the general deterministic model is such that the following equations are satisfied:

$$\text{(1)} \qquad \begin{aligned} \frac{db}{dt} &= -\alpha(b,r,t) \ , \\ \frac{dr}{dt} &= -\beta(b,r,t) \ . \end{aligned}$$

When we view the process stochastically, that is, when $B(t)$ and $R(t)$ are taken to be random variables, the attrition terms $\alpha(b,r,t)$ and $\beta(b,r,t)$ translate into infinitesimal transition probabilities according to the equations:

$$P\{B(t+h) = b-1,\ R(t+h) = r \mid B(t) = b,\ R(t) = r\}$$
$$= \alpha(b,r,t)h + o(h)\ ,$$

(2) $$P\{B(t+h) = b,\ R(t+h) = r-1 \mid B(t) = b,\ R(t) = r\}$$
$$= \beta(b,r,t)h + o(h),$$

$$P\{\text{two or more changes in } (t,t+h)\} = o(h)\ ,$$

and hence

$$P\{B(t+h) = b,\ R(t+h) = r \mid B(t) = b,\ R(t) = r\}$$

$$= 1 - \{\alpha(b,r,t) + \beta(b,r,t)\}h + o(h)\ ,$$

where $\lim_{h\to 0} o(h)/h = 0$.

If we write

$$p_{b,r}(t) = P\{B(t) = b,\ R(t) = r\}\ ,$$

then the forward differential-difference equation governing this process is (there is a corresponding backward equation also)

(3) $$\frac{d}{dt} p_{b,r}(t) = -\{\alpha(b,r,t) + \beta(b,r,t)\}\ p_{b,r}(t)$$
$$+ \alpha(b+1,r,t)\ p_{b+1,r}(t) + \beta(b,r+1,t)\ p_{b,r+1}(t),$$

for $(b,r) \in A = \{(b,r): 0 \le b \le B_0,\ 0 \le r \le R_0\}$, and where

$p_{b,r}(t) \equiv 0$ whenever $(b,r) \notin A$. The initial conditions are

$$p_{B_0,R_0}(0) = 1, \quad p_{b,r}(0) = 0 \quad \text{for} \quad (b,r) \neq (B_0,R_0).$$

As written here the transition probability generators $\alpha(b,r,t)$ and $\beta(b,r,t)$ are any (positive) function of the state of the process (b,r) and any function of time t. Frequently, the models of interest are such that these generators are time independent, that is, $\alpha(b,r,t) = \alpha(b,r)$ and $\beta(b,r,t) = \beta(b,r)$. When this holds, the set of equations (3) is just the bivariate pure death process described by Billard and Kryscio (1977). They then give the solution for $p_{b,r}(t)$ for this equation for any generalized $\alpha(b,r)$ and $\beta(b,r)$ provided that

$$\alpha(b_1,r_1) + \beta(b_1,r_1) \neq \alpha(b_2,r_2) + \beta(b_2,r_2) \quad \text{for} \quad (b_1,r_1) \neq (b_2,r_2). \tag{4}$$

Thus,

$$p_{b,r}(t) = \sum_{m=b}^{B_0} \sum_{w=r}^{R_0} c_1(m,w|B_0,R_0)\, c_2(m,w|b,r) \exp\{d(m,w)t\}, \tag{5}$$

where

$$d(b,r) = -\alpha(b,r) - \beta(b,r)$$

and where $c_1(m,w|B_0,R_0)$ and $c_2(m,w|b,r)$ are given in Billard and Kryscio (1977, eqns. 10-11).

Most processes of practical interest are such that (4) holds naturally. In obtaining their solution, Billard and Kryscio (1977) essentially exploit and utilize a more general theorem given in Severo (1969a). In the event that the restriction (4) does not hold, the solution can still be found by using the Severo theorem directly. Likewise, when the transition generators are functions of time t, direct use of Severo's theorem yields the required solution to (2). In any event, exploitation of the underlying structure which expresses itself as a partitioning scheme as used in Billard and Kryscio (1977) and Billard (1980), has the advantageous effect of reducing considerably the degree of complexity that a first glance at Severo's result suggests is involved.

3. LANCHESTER'S MODEL

The simplest formulation for combat models is that of Lanchester (1914) where it is supposed that losses on each side are proportional to the size of the opposing force. This is the case of "aimed" fire. That is,

$$\alpha(b,r,t) = \alpha(b,r) = \gamma_1 r$$

and

$$\beta(b,r,t) = \beta(b,r) = \gamma_2 b \ .$$

For convenience, we rescale so that $\alpha(b,r) = \lambda r$ and $\beta(b,r) = b$ where now λ is the relative effectiveness of the Red force to the Blue force. Then, the appropriate differential-difference equation governing this process is simply, from (3),

$$(6) \qquad \frac{d}{dt} p_{b,r}(t) = -(\lambda r + b) \ p_{b,r}(t) + b \ p_{b,r+1}(t) + \lambda r \ p_{b+1,r}(t)$$

for $(b,r) \in A$ and where we note that for boundary values (b,r) suitable adjustment is necessary. In matrix form, we may write (6) as

$$\left(\frac{d}{dt} p_{b,r}(t)\right) = \underline{B}(p_{b,r}(t))$$

where $\underline{B}$ is the matrix of coefficients (of transition generators). Thus in the particular case that $B_0 = 4$, $R_0 = 2$ and $\lambda = .8$, $\underline{B}$ is given in Table 1.

The solution to (6) is given by (5) where it is readily verified that

(7a) $c_1(m,w|B_0,R_0)$

$$
= \begin{cases}
(\lambda w)^{B_0-m}/(B_0-m)! \,, & w = R_0, \\[2ex]
\displaystyle\sum_{i_{R_0-w}}^{i_{R_0-w+1}} \cdots \sum_{i_1=1}^{i_2} \frac{\lambda^{B_0-m} R_0^{\,i_1-i_0} (R_0-1)^{i_2-i_1} \cdot \ldots \cdot w^{i_{R_0-w+1}-i_{R_0-w}}}{(B_0 - m + 1 - i_{R_0-w})!} & \\[2ex]
\displaystyle\times \prod_{v=1}^{R_0-w} \left[\frac{(B_0 + 1 - i_v)}{\{B_0-m+1-i_v+\lambda(R_0-w+1-v)\}} \right. & \\[2ex]
\displaystyle\qquad \left. \times \prod_{j_v=i_{v-1}}^{i_v-1} \{B_0-m+1-j_v+\lambda(R_0-w+1-v)\}^{-1} \right], & w < R_0 \,,
\end{cases}
$$

with $i_0 = 1$ and $i_{R_0-w+1} = B_0-m+1$; and

(7b) $c_2(m,w|b,r)$

$$
= \begin{cases}
(-1)^{m-b}(\lambda w)^{m-b}/(m-b)! \,, & r = w \,, \\[2ex]
\displaystyle\sum_{i_{w-1}=1}^{i_{w-r+1}} \cdots \sum_{i_1=1}^{i_2} (-1)^{i_{w-r+1}-i_{w-r}} \frac{\lambda^{m-b} w^{i_1-i_0} (w-1)^{i_2-i_1} \cdot \ldots \cdot r^{i_{w-r+1}-i_{w-r}}}{\{\lambda(w-r) + m - b\}!/\{\lambda(w-r) + i_{w-r}-1\}!} & \\[2ex]
\displaystyle\times \prod_{v=1}^{w-r} \left[\frac{(m-i_v+1)}{(1-v\lambda-i_v)} \prod_{j_v=i_{v-1}}^{i_v-1} \{\lambda(1-v)-j_v\}^{-1} \right], & r < w \,,
\end{cases}
$$

with $i_{w-r+1} = m-b+1$.

In matrix notation, the set of state probabilities may be written as

$$(p_{b,r}(t)) = \underline{C}\ \underline{e}(t) ,$$

where the vector $\underline{e}(t)$ has elements $\{\exp(b_i t)\}$ with b_i being the $i\underline{th}$ diagonal element of $\underline{B}$ and the elements of the matrix $\underline{C}$ being determined from (7). In our example, the corresponding $\underline{C}$ matrix is given in Table 2. Thus, for example,

$$P\{B(t) = 2, R(t) = 2\} = 1.28e^{-5.6t} - 2.56e^{-4.6t} + 1.283e^{-3.6t}.$$

Further details on how to calculate these quantities algorithmically will be discussed in the forthcoming paper dealing with numerical calculations and comparisons generally.

We notice that the underlying structure for both $c_1(\cdot)$ and $c_2(\cdot)$ is very similar, and furthermore within each formula the structure for each i_v term is also similar. This allows computer calculation of these quantities to run without too much difficulty. This effect is especially apparent in the particular case that $\lambda = 1$, that is, when the two forces are equally effective. In this case we find that

$$c_1(m,w|B_0,R_0)\cdot(B_0+R_0-m-w)!$$

$$=\begin{cases} w^{B_0-m}, & w=R_0, \\ \displaystyle\sum_{i_{R_0-w}=1}^{i_{R_0-w+1}} \cdots \sum_{i_1=1}^{i_2} R_0^{\,i_1-i_0}(R_0-1)^{i_2-i_1}\cdot\ldots\cdot w^{i_{R_0-w+1}-i_{R_0-w}} & \\ \qquad \times \displaystyle\prod_{v=1}^{R_0-w}(B_0+1-i_v), & w<R_0; \end{cases}$$

and

$$c_2(m,w|b,r)\cdot(m+w-b-r)!$$

$$=\begin{cases} (-1)^{m-b}\,w^{m-b}, & r=w, \\ \displaystyle\sum_{i_{w-r}=1}^{i_{w-r+1}} \cdots \sum_{i_1=1}^{i_2} (-1)^{m+w-b-r} w^{i_1-i_0}(w-1)^{i_2-i_1}\cdot\ldots\cdot r^{i_{w-r+1}-i_{w-1}} & \\ \qquad \times \displaystyle\prod_{r=1}^{w-r}(m+1-i_v), & r<w. \end{cases}$$

Gye and Lewis (1976) have argued that it is quite reasonable that in the Battle of Trafalgar, $\lambda = 1$. Other such situations can be easily visualized.

Once expressions for $p_{b,r}(t)$ have been determined, other quantities of interest can be obtained. Suppose we are interested in the event that in time t the Blue force loses x units while the Red force loses none. Substitution in (5) and (7) yields the result

(8) $P\{B(t) = B_0 - x,\ R(t) = R_0\}$

$$= p_{B_0-x,R_0}(t) = \exp[-(B_0-x+R_0\lambda)t]\ (\lambda R_0)^x(1-e^{-t})^x/x!\ ,$$

$$x = 0,\ \ldots\ ,\ B_0\ .$$

Likewise, if we are concerned with the probability that Red loses no units (regardless of the number of loses on the Blue side) up to time t, we have

(9) $$P\{R(t) = R_0\} = \sum_{x=0}^{B_0} p_{B_0-x,R_0}(t)$$

which can be determined from (8).

These results (8) and (9) could be likened to the model described by Gaver (1979) in which a force of initial size B_0 attacks a bastion or stronghold of size R_0. Gaver assumed that the Red's stronghold is sufficient to guarantee no loss of Red units. He then considers the size of the Blue force at time t using always the deterministic approach. Gaver looks at two cases. One case assumes that the attrition rate of the Blue force is a result of unaimed fire by the Red force. Therefore, in the terminology of the present paper, the (stochastic) transition rate is

$$\alpha(b,r,t) = \rho_u(R_0/B_0)b,$$

where ρ_u is the attrition parameter. In the other case, Red employs aimed fire and hence the combat is modelled so that

$$\alpha(b,r,t) = \rho_a b$$

for attrition parameter ρ_a. In either case, Gaver's process is slightly different from the Lanchester results (8) and (9) since, although the quantity of interest includes the fact of no loses to the Red force, it is nevertheless possible probabilistically that the Reds do lose some units by attrition. In Gaver's model it is assumed the Reds cannot lose any units whatsoever.

Returning to the Lanchester model, we can easily obtain the expected number of units lost by time t by the Blue force given that the Red force has no loses. Let this random variable be denoted by $X(t)$. Then,

$$E\{X(t)\} = \sum_{x=0}^{B_0} x \, P\{B(t) = B_0 - x | R(t) = R_0\}$$

$$= \sum_{x=0}^{B_0} x \, p_{B_0-x,R_0}(t) / \sum_{x=0}^{B_0} p_{B_0-x,R_0}(t)$$

$$= \lambda R_0 (1-e^{-t}) \; S(B_0-1)/S(B_0) \; ,$$

where

$$S(p) = \sum_{y=0}^{p} \{\lambda R_0 (1-e^{-t}) e^{t}\}^y / y! \; .$$

If B_0 is sufficiently large so that $S(B_0) \approx S(B_0-1)$, we have

$$E\{X(t)\} \approx \lambda R_0 (1-e^{-t}) \; .$$

Hence, the eventual expected number of loses on the Blue side is

$$\lim_{t \to \infty} E\{X(t)\} = \lambda R_0 \; .$$

This result is what we would expect intuitively.

We can relate the stochastic and deterministic models through the expectations since the attrition rates are linear functions of the variables involved. Thus, if we multiply (6) throughout by b and sum over all (b,r) values, we have

$$\frac{d}{dt} E\{B(t)\} = -\lambda E\{R(t)\} .$$

Likewise, multiplying by r and summing over (b,r) gives

$$\frac{d}{dt} E\{R(t)\} = - E\{B(t)\} .$$

Comparison with the deterministic equations shows that the expected values of the stochastic variables equals the solution of the deterministic equations. We note this correspondence does not necessarily hold when the attrition rates become non-linear in b and/or r.

4. OTHER COMBAT MODELS

In this section we briefly describe how other combat models can be expressed stochastically. There have been many models proposed in the literature. An extensive and exhaustive review of such models has been made by Taylor (1978) to which we refer the reader for further details, elaboration, and justification. These models have by and large been studied from a deterministic viewpoint only. We confine outselves here to establishing certain stochastic analogues of some of these processes. Specifically, we obtain the appropriate format for the transition probability generators $\alpha(b,r,t)$ and $\beta(b,r,t)$.

Once the generators have been established, it is then a relatively simple and direct procedure to solve the corresponding differential-difference equations as discussed earlier in Section 2. Obviously the simpler the form for $\alpha(b,r,t)$ and $\beta(b,r,t)$, the simpler the resulting solutions will be, such as we saw in the previous section for the original Lanchester model. Quite clearly the reduction achieved depends on the actual structure of these generators. When $\alpha(b,r,t)$ and $\beta(b,r,t)$ are linear functions of b or r, such reduction will be substantial and nice. However, even the most complicated cases can still be solved since use of the Severo theorem is completely general without any confining restrictions as far as our models are concerned.

Taylor and Brown (1976) establish what they call "variable-coefficient Lanchester-type equations of modern warfare"

where the attrition rate for each side is time-dependent. By analogy, we find the transition probability generators to be given by

$$(10) \qquad \alpha(b,r,t) = a(t)r \quad \text{and} \quad \beta(b,r,t) = b(t)b \ .$$

If $a(t) = \gamma_1 h(t)$ and $b(t) = \gamma_2 h(t)$, rescaling of time from t to $h(t)$ produces generators

$$\alpha(b,r,t) = \gamma_1 r \quad \text{and} \quad \beta(b,r,t) = \gamma_2 b \ ,$$

which is the Lanchester model of Section 3.

Range-dependent attrition rates were introduced by Bonder (1967). Here, we have

$$\alpha(b,r,t) = -\alpha(d)r \quad \text{and} \quad \beta(b,r,t) = -\beta(d)b \ ,$$

where $\alpha(d)$ and $\beta(d)$ are the attrition rates for the Blue and Red forces, respectively, dependent on the range or distance between the two forces. In some cases of tactical interest, $\alpha(d)$ and $\beta(d)$ may be represented by the function

$$\alpha(d) = \begin{cases} \rho_1(1 - d/d_2)^{\mu_1} \ , & 0 \leq d \leq d_2 \ , \\ 0 \ , & d_2 < d \ , \end{cases}$$

and

$$\beta(d) = \begin{cases} \rho_2(1 - d/d_1)^{\mu_2}, & 0 \leq d \leq d_1 , \\ 0 , & d_1 < d , \end{cases}$$

where d_1 and d_2 are the maximum effective ranges of the Blue and Red forces, respectively, and where ρ_1, ρ_2, μ_1 and μ_2 are appropriate nonzero constants. If at $t = 0$, $d \leq M$ in (d_1, d_2), that is, when combat begins both forces are within weaponary range of each other, and if there is no retreat, then this process reduces to the Lanchester model of Section 3 where now

$$\gamma_i = \rho_i(1 - d/d_{3-i})^{\mu_i}, \quad i = 1,2 .$$

However, it is equally (perhaps more) reasonable to assume that the range parameter d is itself time dependent. Bonder (1967) takes

$$d \equiv d(t) = d_0 - vt$$

where d_0 is the initial range at $t = 0$ and v is the constant attack (closure) speed. Thus, the attrition rates $\alpha(d)$ and $\beta(d)$ are now time dependent.

The range-dependent attrition rates of Bonder are a particular case of a more general power attrition rate class of models discussed by Taylor and Brown (1976) and Taylor and Comstock (1977). This general class is such that the transition probability generators assume the form $\alpha(b,r,t) = k_1(t + C)^{\mu_1}$ and

$\beta(b,r,t) = k_2(t + C + A)^{\mu_2}$, where $A \geq 0$ is called the offset parameter and $C \geq 0$ is called the starting parameter. This accommodates the situation in which the forces have different maximum effective ranges and/or the battle begins within those maximum ranges. Taylor and Comstock (1977) give the solution to the deterministic model when there is no offset that is, $A = 0$, as well as for $A > 0$ when $\mu_1 = \mu_2 = 1$ and $\mu_1 = 1$, $\mu_2 = 2$. We note that the stochastic solution can be obtained for all μ_1 and μ_2.

A distinguishing characteristic of all the models considered so for is that the attrition rate of each force is proportional to the size of the other force and in no way depends on its own size. It is quite reasonable to expect that attrition could be proportional to the size of both opposing forces. This is especially so far unaimed fire (but can be equally argued for aimed fire) when it is realized that the probability that a unit on the Red force, say, will kill one unit (out of the remaining $B(t)$ units) on the Blue force, increases (decreases) as the size $B(t)$ increases (decreases) since there are more (fewer) units on which his fire may fall. In this context, the appropriate transition probability generators in a time independent situation would be $\alpha(b,r,t) = \gamma_1 br$ and $\beta(b,r,t) = \gamma_2 br$. We note that here the generators are time independent so that the solution (5) holds directly. However, there is effectively a time dependency in that the actual size of each force is of course a function of time.

This model is mathematically equivalent to the Weiss (1963) predator prey model whose solution is given in Billard and Kryscio (1977).

Let us now consider the situation in which one or both sides has a supporting system which itself is not subject to attrition. Taylor and Parry (1975) considered a particular case of such a process which when presented as a stochastic process will have transition probability generators

$$\alpha(b,r,t) = -a(t)r - b_1(t)b ,$$

and

$$\beta(b,r,t) = -b(t)b - a_1(t)r ,$$

where $a(t)$ and $b(t)$ have the same interpretation as in (10) and where $a_1(t) \geq 0$ and $b_1(t) \geq 0$ represent the attrition coefficients on the Red, and Blue, forces due to the supporting fire of the Blue, and Red, forces, respectively. Other situations come readily to mind such as a model in which the attrition from the supporting fire is also dependent on the size of its target (opposing force). Then, we would have

$$\alpha(b,r,t) = -a(t)r - b_1(t)br ,$$

with $\beta(b,r,t)$ similarly defined if the same conditions prevail for the Red side.

Finally, Helmbold (1965) concerned himself with the situation in which grossly unequal force sizes are in combat. His modification of the Lanchester equations, when viewed stochastically, gives us

$$\alpha(b,r,t) = -a(t)\ h(b/r)r$$

and

$$\beta(b,r,t) = -b(t)\ h(r/b)b\ ,$$

where $h(z)$ is interpreted as the effectiveness-modification factor. Helmbold imposed some conditions on the $h(z)$. However, we note that for the stochastic model we are able to find the appropriate conditions without his restrictions

Quite clearly the concept of supporting fire can be combined with Helmbold's model. This has been done by Taylor (1976) in the particular case that $h(z) = z^c$ for some constant $c \geq 0$.

5. BIRTH AND DEATH MODELS

One feature of the models discussed to date is that only losses due to attrition occur to the combat forces, that is, they are pure death processes. However, there are some situations which may be adequately modelled as a birth and death process as, for example, when reinforcements are permitted. If the maximum number of possible reinforcements is known in advance, then we have a bounded birth and death process.

As before, let $B(t)$ and $R(t)$ represent the size of the Blue and Red forces at time t, respectively; and let $B_0 = B(0)$ and $R_0 = R(0)$. Suppose that at time t there remain $B^*(t)$ and $R^*(t)$ Blue and Red reinforcements, respectively and suppose $B^*(0) = B_0^*$ and $R^*(0) = R_0^*$, that is, B_0^* and R_0^* represent the maximum possible reinforcements available. Thus, after time t, $B_0^* - B^*(t)$ actual reinforcements have been added to the Blue combat force, with $R_0^* - R_0^*(t)$ similarly defined for the Red side. Let $B_0^* + B_0 = B$ and $R_0^* + R_0 = R$. Let $\underline{X}(t) = (B^*(t), B(t), R^*(t), R(t))$ with realization $\underline{x} = (b^*,b,r^*,r)$. Then, for a completely general model, we have the following infinitesimal transition probabilities (corresponding to (2))

$$P\{\underline{X}(t+h) = (b^*-1,b+1,r^*,r)\,|\,\underline{X}(t) = \underline{x}\} = \lambda_1(\underline{x},t)h + o(h) \ ,$$

$$P\{\underline{X}(t+h) = (b^*,b,r^*-1,r+1)\,|\,\underline{X}(t) = \underline{x}\} = \lambda_2(\underline{x},t)h + o(h) \ ,$$

$$(11) \quad P\{\underline{X}(t+h) = (b^*,b-1,r^*,r)\,|\,\underline{X}(t) = \underline{x}\} = \mu_1(\underline{x},t)h + o(h) \ ,$$

$$P\{\underline{X}(t+h) = (b^*,b,r^*,r-1)\,|\,\underline{X}(t) = \underline{x}\} = \mu_2(\underline{x},t)h + o(h) \ ,$$

$$P\{\text{two or more changes in } (t,t+h)\} = o(h) \ ,$$

and hence

$$P\{\underline{X}(t+h) = \underline{x} | \underline{X}(t) = \underline{x}\} = 1 - \sum_{i=1}^{2} \{\lambda_i(\underline{x},t) + \mu_i(x,t)\}h + o(h) .$$

If we write

$$p(\underline{x},t) = P\{\underline{X}(t) = \underline{x}\},$$

the differential-difference equation governing the process is (compare (3))

$$\text{(12)} \quad \frac{d}{dt} p(\underline{x},t) = - \sum_{i=1}^{2} \{\lambda_i(\underline{x},t) + \mu_i(\underline{x},t)\} \, p(\underline{x},t)$$

$$+ \lambda_1(b^*+1,b-1,r^*,r,t) \, p(b^*+1,b-1,r^*,r,t)$$

$$+ \lambda_2(b^*,b,r^*+1,r-1,t) \, p(b^*,b,r^*+1,r-1,t)$$

$$+ \mu_1(b^*,b+1,r^*,r) \, p(b^*,b+1,r^*,r,t)$$

$$+ \mu_2(b^*,b,r^*,r+1) \, p(b^*,b,r^*,r+1,t) ,$$

for

$$\underline{x} \in \mathcal{B} = \left\{ \begin{array}{l} \underline{x}: 0 \le b^* \le B^*, \ 0 \le r^* \le R_0^* \\ \quad 0 \le b \le B, \ 0 \le r \le R \end{array} \right\} ,$$

and where $p(\underline{x},t) \equiv 0$ whenever $\underline{x} \notin \mathcal{B}$. The initial conditions are

$$p(B_0^*,B_0,R_0^*,R_0,0) = 1 , \quad p(\underline{x},0) = 0 \quad \text{for} \quad \underline{x} \ne (B_0^*,B_0,R_0^*,R_0) .$$

The equation (12) can be solved by combining the techniques of Severo (1969a,1969b). When the transition generators are time independent, that is, $\lambda_i(\underline{x},t) = \lambda_i(\underline{x})$ and $\mu_i(\underline{x},t) = \mu_i(\underline{x})$, $i = 1,2$, an explicit solution for any arbitrary $\lambda_i(\underline{x})$ and $\mu_i(\underline{x})$, $i = 1,2$, is given in Billard (1980).

The $\mu_i(x,t)$, $i = 1,2$, terms give the attrition rates and typically we would expect them to assume the same forms as the $\alpha(b,r,t)$ and $\beta(b,r,t)$, respectively, presented earlier. The terms $\lambda_i(\underline{x},t)$, $i = 1,2$, represent the rates that reinforcements are added. Though written generally here, it is reasonable to expect that for models of practical interest, these generators will be a function of the size of the combat force itself. Thus, for example, the case where more reinforcements are brought in when the combat force itself becomes small could be reflected by a transition generator

$$\lambda_1(\underline{x},t) = \lambda(t)/b \ ,$$

or simply

$$\lambda_1(\underline{x},t) = \lambda/b \ .$$

Recently, Gaver (1979) investigated the problem of information flow. In his model, individual units start off with unaimed fire but then as information about the opposing force is gathered the fire becomes aimed fire. Thus, in the $\underline{X}(t)$ notation used above, $B^*(t)$ is the number of Blue units still using unaimed

fire at time t, $B(t)$ is the number of Blue units using aimed fire at time t, and $R^*(t)$ and $R(t)$ are the corresponding quantities for the Red force. If all units improve their fire to aimed fire before they themselves are killed, we have exactly the situation modelled in (11) and (12). However, it is more realistic to assume that some of the units never learn from the information gathered, thus remaining with unaimed fire, and will themselves be lost by attrition. To model this situation, the infinitesimal transition probabilities (11) still apply in addition to

$$P\{\underline{X}(t+h) = (b^*-1,b,r^*,r)|\underline{X}(t) = \underline{x}\} = \gamma_1(\underline{x},t)h + o(h) ,$$

and

$$P\{\underline{X}(t+h) = (b^*,b,r^*-1,r)|\underline{X}(t) = \underline{x}\} = \gamma_2(\underline{x},t)h + o(h) ,$$

and where now

$$P\{\underline{X}(t+h) = \underline{x}|\underline{X}(t) = \underline{x}\} = 1 - \sum_{i=1}^{2} \{\lambda_i(\underline{x},t) + \mu_i(\underline{x},t) + \gamma_i(\underline{x},t)\}h + o(h).$$

Thus, for each $i = 1,2$, the $\lambda_i(\underline{x},t)$ represent the information flow, the $\mu_i(\underline{x},t)$ represent the loss of aimed units from attrition, and the $\gamma_i(\underline{x},t)$ represent the loss of unaimed units from attrition. Suitable adjustment of equation (12) will give us the appropriate differential-difference equation for the process. This latter equation an then be solved using the method of Severo (1969a,1969b).

It is now possible to establish Gaver's (1979) model for information flow under mutual attrition in a stochastic framework. Thus, we have

$$\lambda_1(\underline{x},t) = \alpha_{ua}b^* \quad \text{and} \quad \lambda_2(\underline{x},t) = \beta_{ua}r^* ,$$

where α_{ua} and β_{ua} represent the rate of conversion from an unaimed capacity to aimed capacity for the Blue and Red forces, respectively;

$$\mu_1(\underline{x},t) = \rho_{ua}r^*b/B + \rho_{aa}rb/(b + b^*)$$

and

$$\mu_2(\underline{x},t) = \gamma_{ua}b^*r/R + \gamma_{aa}br/(r + r^*) ,$$

where ρ_{ua} represents the attrition rate of the unaimed Red forces against the aimed Blue forces, ρ_{aa} represents the attrition rate of the aimed Red forces against the aimed Blue forces, and γ_{ua} and γ_{aa} are the corresponding attrition rates of the Blue forces against the Red forces; and

$$\gamma_1(\underline{x},t) = \rho_{uu}r^*b^*/B + \rho_{au}rb^*/(b + b^*)$$

and

$$\gamma_2(\underline{x},t) = \gamma_{uu}b^*r^*/R + \gamma_{au}br^*/(r + r^*) ,$$

where ρ_{uu} and ρ_{au} represent the rate of attrition to the unaimed Blue forces due to the unaimed and aimed fire, respectively, from the Red forces, and γ_{uu} and γ_{au} are similarly defined attrition rates of the Red forces.

6. COMBAT DURATION TIME

The previous sections have been concerned primarily with techniques for obtaining the state probabilities of the underlying distribution. A knowledge of these probabilities allows most other quantities of interest to be derived. One such quantity is the combat duration time or the expected length of time for the battle. While this can certainly be obtained from our earlier results, we present here a relatively easy straightforward but recursive method for its derivation.

Suppose that at $t = 0$, the Blue force has B_0 units and the red force has R_0 units. Let us assume that the combat ends when the Blue force is reduced to a size of B' units and/or the Red force is reduced to a size of R' units. Hence, the permitted state space for (b,r) is now $A' = \{(b,r): B' \leq b \leq B_0, R' \leq r \leq R_0\}$ instead of the A used in the previous section. (The adjustment from A to A' in the results given earlier is trivial.) Let $T(b,r)$ be the expected combat duration time from the time that there are b Blue units and r Red units remaining. Then, $T(B_0,R_0)$ will be the overall combat duration time. For ease of illustration, let us suppose we have the simple Lanchester combat model where $\alpha(b,r,t) = \lambda r$ and $\beta(b,r,t) = b$. More general models are treated analogously.

We first recall that the basic underlying process is Markovian. From (2), and basic properties of Markov processes, we see that when the system moves from the state (b,r) it moves

to the state $(b-1,r)$ with probability $\lambda r/(b+\lambda r)$ or to the state $(b,r-1)$ with probability $b/(b+\lambda r)$. Finally, before leaving the state (b,r) it will have stayed there for an average time of $1/(b+\lambda r)$. Therefore, we can write

$$(13) \quad T(b,r) = 1/(b+\lambda r) + \lambda r/(b+\lambda r)\, T(b-1,r) + b/(b+\lambda r)\, T(b,r-1),$$

for $(b,r) \in A'$. In (13), $T(b,r) \equiv 0$ for $(b,r) \notin A'$. Clearly then, by starting at $(b,r) = (B',R')$ and proceeding with (B',r), $r = R'+1, \ldots, R_0$, returning to $(B'+1,r)$, $r = R',\ldots,R_0$, and so on to (B_0,r), $r = R',\ldots,R_0$, we can find $T(B_0,R_0)$. Note that in fact we have a matrix of $T(b,r)$ values which give us the duration time from any intermediate stage (b,r) in addition to the overall duration $T(B_0,R_0)$.

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TABLE 1

Matrix of coefficients (of transition generators) $\underline{B}$ when $B_0 = 4$, $R_0 = 2$, $\lambda = .8$

$$
B = \begin{array}{c}
\left[\begin{array}{ccccccccccccccc}
-5.6 & & & & & & & & & & & & & & \\
4 & -4.8 & & & & & & & & & & & & & \\
 & 4 & 0 & & & & & & & & & & & & \\
1.6 & & & -4.6 & & & & & & & & & & & \\
 & .8 & & 3 & -3.8 & & & & & & & & & & \\
 & & 0 & & 3 & 0 & & & & & & & & & \\
 & & & 1.6 & & & -3.6 & & & & & & & & \\
 & & & & .8 & & 2 & -2.8 & & & & & & & \\
 & & & & & 0 & & 2 & 0 & & & & & & \\
 & & & & & & 1.6 & & & -2.6 & & & & & \\
 & & & & & & & .8 & & 1 & -1.8 & & & & \\
 & & & & & & & & 0 & & 1 & 0 & & & \\
 & & & & & & & & & 1.6 & & & 0 & & \\
 & & & & & & & & & & .8 & & 0 & 0 & \\
 & & & & & & & & & & & 0 & & 0 & 0
\end{array}\right]
\end{array}
\begin{array}{c}
(b,r) \\
4\ 2 \\ 4\ 1 \\ 4\ 0 \\ 3\ 2 \\ 3\ 1 \\ 3\ 0 \\ 2\ 2 \\ 2\ 1 \\ 2\ 0 \\ 1\ 2 \\ 1\ 1 \\ 1\ 0 \\ 0\ 2 \\ 0\ 1 \\ 0\ 0
\end{array}
$$

TABLE 2

Coefficients of "solution" matrix $\underline{C}$ when $B_0 = 4$, $R_0 = 2$, $\lambda = .8$

$$
\underline{C} = \begin{bmatrix}
1 & & & & & & & & & & & & & & \\
-5.0 & 5.0 & & & & & & & & & & & & & \\
3.571 & -4.167 & .596 & & & & & & & & & & & & \\
-1.6 & 0 & 0 & 1.6 & & & & & & & & & & & \\
4.889 & -4.0 & 0 & -6.0 & 5.111 & & & & & & & & & & \\
-2.619 & 2.50 & 0 & 3.913 & -4.035 & .241 & & & & & & & & & \\
1.28 & 0 & 0 & -2.56 & 0 & 0 & 1.28 & & & & & & & & \\
-2.311 & 1.60 & 0 & 5.511 & -4.089 & 0 & -3.20 & 2.489 & & & & & & & \\
.825 & -.667 & 0 & -2.396 & 2.152 & 0 & 1.778 & -1.778 & .086 & & & & & & \\
-.683 & 0 & 0 & 2.048 & 0 & 0 & -2.048 & 0 & 0 & .683 & & & & & \\
.666 & -.427 & 0 & -2.306 & 1.636 & 0 & 2.560 & -1.991 & 0 & -.854 & .716 & & & & \\
-.119 & .089 & 0 & .501 & -.431 & 0 & -.711 & .711 & 0 & .329 & -.398 & .029 & & & \\
.195 & 0 & 0 & -.712 & 0 & 0 & .910 & 0 & 0 & -.420 & 0 & 0 & .027 & & \\
-.095 & .071 & 0 & .401 & -3.443 & 0 & -0.566 & .569 & 0 & 2.627 & -.318 & 0 & 0 & .754 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

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